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FERMAT´S LAST THEOREM: A GEOMETRIC PROOF

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ABSTRACT

We have discovered a truly marvelous proof, which this article can contain. Indeed, this article has the goal of present a solution to Fermat s Last Theorem. It is a Diophantine equation that has enchanted all subjects of the mathematical world for centuries due to your frugality. So near, and yet so distant. The equation was and still a enigma. By the year of 1993, the British mathematician Andrew Willes presented to the world a solution. But, we believe in not several ways, but,at least, in more than one way that can solve it. Or, yet, we are trying to reach the same point, through a different point of view. Unlikely, but possible. A simple and straight resolution; moreover, compatible with the seventeenth Fermat´s environment, time and geniality. After all, as Da Vinci stated centuries ago, "simplicity isthe ultimate sophistication".

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INTRODUCTION

According to Pierre Fermat, the equation $a^n = b^n + c^n$ with $\{n > 2 | n \in \mathbb{N}\}$ has no solution in which $(a, b, c) \in N$. This is the Fermat´s last theorem. It was one of the last enigmas of the mathematical in world and persisted for more than three centuries. By the year of 1993, Fermat´s Last Theorem was mathematically demonstrated by Andrew Willes, in a paper, initially with more than two hundreds pages. The British mathematician used sophisticated concepts and tools that were not known at Fermat´s time. In this paper, we are going to use plane geometry, trigonometric functions and algebraic principles to demonstrate the truthiness of the theorem.

Argument and Proof

Equation notation

Suppose the equation

$$a^n = b^n + c^n \quad (1)$$

Then, by the statement that the whole is greater than the parts, we have $a^n > b^n$and $a^n > c^n$. That is to say that $a > b$ and $a > c$. Thereby, the three possible ways that a is related to the sum of $(b + c)$ are:

$$a < b + c\,, \quad (2)$$
$$a = b + c\,, \quad (3)$$
$$a > b + c\,. \quad (4)$$

Whence, (2), (3) and (4) imply, respectively:

$$a^n < (b + c)^n, a^n = (b + c)^n \text{ and } a^n > (b + c)^n$$

Yet, developing the binomial $(b + c)^n$, as result we have:

$$(b + c)^n = b^n + \binom{n}{1} b^{n-1} c + \cdots + \binom{n}{n-1} b\, c^{n-1} + c^n$$

Considering the binomial´s development; and, comparing with (4), it will always have $a^n > b^n + c^n$, because, $a^n > (b + c)^n$. So, (1) will never find solution in (4). Yet, comparing the binomial´s development with (2) and (3), these remains possible as solution of (1).

So, there will be (1), if and only if:

$$a > b\,, \quad a > c \quad \text{and} \quad a \le b + c\,.$$

Argument: If $a > b$, $a > c$ and $a \le b + c$, then, it is to say geometrically that (a, b, c) satisfies the condition of existence of a triangle, according to the triangle inequality theorem. Yet, the only two possibles orders for the tern (a, b, c) are the following:

$a > b > c$ or $a > c > b$

For proof and argument purpose, we are going to consider $a > c >$ b. At this time, we register that either one of the orders chosen will not neither mislead the proof development, nor the final result of it. So, *a*, *b* and *c* will stand for the sides of a triangle, and also as its scalar measures. Because of that, we have the geometrical argument below.

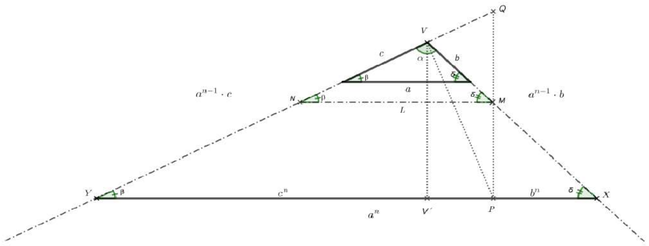

Figure 1.

Then, we make the triangle $\triangle(a, b, c)$, as shown at the Fig. 1. Next, b and c are extend until it fits the segment a^n; and, $a \parallel a^n$. As a result of it, two similar triangles with parallel bases a and a^n are formed. Thus, by the proportionality statement, the similarity ratio of these two triangles is a^{n-1}. Yet again, by the proportionality statement, it is possible to obtain the other two sides of the bigger triangle of base a^n: $a^{n-1} \cdot b$ and $a^{n-1} \cdot c$.

On the other hand, if $a^n = b^n + c^n$ (1), there will be a point *P* which separates b^n from c^n on a^n. And, *P* could or could not be coincident with the orthogonal projection of *V*, that is *V´*, on a^n.

Proof

Hypotheses and resolutions

Suppose exists $a^n = b^n + c^n$ (1) with $(a, b, c) \in \mathbb{N}$ and $\{n > 2 | n \in \mathbb{N}\}$; and *P*, which separates b^n from c^n on a^n, could or not to be coincident with the orthogonal projection of *V: V´*. From it, there will be three situations.

Situation – 1

$b^n > a^{n-1} \cdot b \cdot \cos\delta$ (5) and $c^n < a^{n-1} \cdot c \cdot \cos\beta$ (6).

If and only if, *P* is on the left of *V´*.

Situation – 2

$b^n = a^{n-1} \cdot b \cdot \cos\delta$ (7) and $c^n = a^{n-1} \cdot c \cdot \cos\beta$ (8).

If and only if, *P* is coincident with *V´*.

Situation – 3

$b^n < a^{n-1} \cdot b \cdot \cos\delta$ (9) and $c^n > a^{n-1} \cdot c \cdot \cos\beta$ (10).

If and only if, *P* is on the right of *V´*.

Regardless of the presented situations, we will have:

$$\cos\delta = \frac{a^2-(c^2-b^2)}{2a \cdot b} \quad \text{and} \quad \cos\beta = \frac{a^2+(c^2-b^2)}{2a \cdot c}$$

And, there they are the demonstrations for each situation.

Proof: situation – 1.

From (5) and (6):
$b^n > a^{n-1} \cdot b \cdot \cos\delta$ (5) and $c^n < a^{n-1} \cdot c \cdot \cos\beta$ (6).

If $\cos\delta$ and $\cos\beta$ are replaced in (5) and (6), it results:

$$b^n > \frac{a^{n-2}\,[a^2-(c^2-b^2)]}{2} \text{ and } c^n < \frac{a^{n-2}\,[a^2+(c^2-b^2)]}{2},$$

or

$$-b^n < -\frac{a^{n-2}\,[a^2-(c^2-b^2)]}{2} \qquad (11)$$

and

$$c^n < \frac{a^{n-2}\,[a^2+(c^2-b^2)]}{2} \qquad (12)$$

Whence, summing (11) and (12):

$$c^n - b^n < a^{n-2} \cdot (c^2 - b^2) \quad \text{or} \quad a^{n-2} > \frac{(c^n-b^n)}{(c^2-b^2)} \qquad (13)$$

Next, effecting the division on the second member of inequality (13), we have:

$$a^{n-2} > c^{n-2} + c^{n-4} \cdot b^2 + \frac{c^{n-4} \cdot b^4 - b^n}{c^2-b^2} \qquad (14)$$

And, multiplying both members of the inequality above times a^2 :

$$a^n > a^2 \cdot c^{n-2} + a^2 \cdot c^{n-4} \cdot b^2 + a^2 \cdot \frac{c^{n-4} \cdot b^4 - b^n}{c^2-b^2} \qquad (14)$$

From (14), and considering $a^2 \cdot c^{n-2} > c^n$, $a^2 \cdot c^{n-4} \cdot b^2 > b^n$ where $a > c >$ b; hence, $a^n > b^n + c^n$.

So, $a^n > b^n + c^n$ whenever the point *P* (yet again, that separates b^n from c^n) is on the left side of *V´*. It is proved the nonexistence of $a^n = b^n + c^n$ (1) for the situation 1, whenever $n > 2$.

Proof: situation – 2

From (7) and (8):

$b^n = a^{n-1} \cdot b \cdot \cos\delta$ (7) and $c^n = a^{n-1} \cdot c \cdot \cos\beta$ (8)

And, by subtracting $c^n - b^n$, it results:

$$c^n - b^n = a^{n-1}\,(c \cdot \cos\beta - b \cdot \cos\delta) .$$

Replacing the cosines, we have:

$$c^n - b^n = a^{n-1} \left[c \cdot \frac{a^2+(c^2-b^2)}{2a \cdot c} - b \cdot \frac{a^2-(c^2-b^2)}{2a \cdot b}\right] .$$

Thus,

$$a^{n-2} = \frac{c^n-b^n}{c^2-b^2} .$$

Next, effecting the division on the second member of the equation above, and multiplying both terms times a^2 :

$$a^n = a^2 \cdot c^{n-2} + a^2 \cdot c^{n-4} \cdot b^2 + a^2 \cdot \frac{c^{n-4} \cdot b^4 - b^n}{c^2-b^2} .$$

This result is similar to the one from situation – 1, inequality (14). Obviously, switching the signal from ">" to the "="; since, this result is an equation, and there is an inequality. But, if $a^2 \cdot c^{n-2} > c^n$ and $a^2 \cdot c^{n-4} \cdot b^2 > b^n$; then, it is to be concluded that $a^n > b^n + c^n$, every time that *P* coincides with *V´*.

Nevertheless, remains another key issue: the assumption that$(a, b, c) \in \mathbb{N}$. Regarding this issue, we can prove that a is note integer.

Returning to (7) and (8), if we replace the cosines on these equations, it results:

$$b^n = a^{n-1} \cdot b \cdot \left[\frac{a^2-(c^2-b^2)}{2a \cdot b}\right] \quad \text{and} \quad c^n = a^{n-1} \cdot c \cdot \left[\frac{a^2+(c^2-b^2)}{2a \cdot c}\right]$$

whence,

$b^n = \frac{a^{n-2}\,[a^2 - (c^2 - b^2)]}{2}$ and $c^n = \frac{a^{n-2}\,[a^2 + (c^2 - b^2)]}{2}$

Thus,

$$a^n - (c^2 - b^2)a^{n-2} - 2b^n = 0 \qquad (15)$$

and

$$a^n + (c^2 - b^2)a^{n-2} - 2c^n = 0 \qquad (16)$$

Now, considering (15) and (16) as polynomial equations of degree *n* in a, with integers an non-zero coefficients; then, none of them will have integer roots. It happens, because, from the hypotheses we have (a, b, c) coprimes pairwise; and, thereby, a will not divide neither $2b^n$, nor $2c^n$. As consequence, it is proved that $a \notin \mathbb{N}$, which is against the hypotheses; where $(a, b, c, n) \in \mathbb{N}$.

So, it is proved the nonexistence of $a^n = b^n + c^n$ (1) for the situation – 2, whenever $n > 2$.

Proof: situation – 3

Once again, returning to Fig. 1, consider the similar triangles $\triangle(V, X, Y) \approx \triangle(V, M, N)$, from where:

$\frac{\overline{VM}}{\overline{VX}} = \frac{\overline{MN}}{\overline{XY}}$ or $\frac{\overline{VX} - \overline{XM}}{\overline{VX}} = \frac{\overline{MN}}{\overline{XY}}$

But, as

$\cos\delta = \frac{\overline{XP}}{\overline{XM}}$ and $\overline{XM} = \frac{\overline{XP}}{\cos\delta}$,

so,

$$\frac{\overline{VX} - \frac{\overline{XP}}{\cos\delta}}{\overline{VX}} = \frac{\overline{MN}}{\overline{XY}} \qquad (17)$$

However,

$\overline{VX} = a^{n-1} \cdot b$, $\overline{XP} = b^n$, $\overline{MN} = L$ and $\overline{XY} = a^n$

Hence, (17) can be written as:

$$\frac{a^{n-1} \cdot b - \frac{b^n}{\cos\delta}}{a^{n-1} \cdot b} = \frac{L}{a^n} \quad \text{or} \quad \frac{a^{n-1} \cdot b \cdot \cos\delta - b^n}{a^{n-1} \cdot b \cdot \cos\delta} = \frac{L}{a^n}$$

Then,

$$L = \frac{a^n \cdot \cos\delta - \ \cdot b^{n-1}}{\cos\delta} \qquad (18)$$

Therefore, returning to Fig. 1, and now considering the similar triangles $\triangle(Q, P, Y) \approx \triangle(Q, M, N)$, we have:

$\frac{\overline{QM}}{\overline{QP}} = \frac{\overline{MN}}{\overline{PY}}$ or $\frac{\overline{QP} - \overline{PM}}{\overline{QP}} = \frac{L}{\overline{PY}}$.

But, as

$\tan\beta = \frac{\overline{QP}}{\overline{PY}}$ or $\overline{QP} = \overline{PY} \cdot \tan\beta$ and $\tan\delta = \frac{\overline{PM}}{\overline{PX}}$ or $\overline{PM} = \overline{PX} \cdot \tan\delta$,

then

$$\frac{\overline{PY} \cdot \tan\beta - \overline{\quad} \cdot \tan\delta}{\overline{PY} \cdot \tan\beta} = \frac{L}{\overline{PY}},$$

So,

$$\frac{c^n \cdot \tan\beta - \ ^n \cdot \tan\delta}{c^n \cdot \tan\beta} = \frac{L}{c^n},$$

whence

$$L = \frac{c^n \cdot \tan\beta - \ ^n \cdot \tan\delta}{\tan\beta} \qquad (19).$$

Continuing into the proof development, we compare (18) and (19), and it results:

$$\frac{a^n \cdot \cos\delta - a \cdot b^{n-1}}{\cos\delta} = \frac{c^n \cdot \tan\beta - \ ^n \cdot \tan\delta}{\tan\beta},$$

And, when we substitute $\tan\beta$ and $\tan\delta$ we come to:

$$\frac{a^n \cdot \cos\delta - \ \cdot b^{n-1}}{\cos\delta} = \frac{\frac{c^n \cdot \sin\beta}{\cos\beta} - \frac{b^n \cdot \sin\delta}{\cos\delta}}{\frac{\sin\beta}{\cos\beta}},$$

or

$$\frac{a^n \cdot \cos\delta - \ \cdot b^{n-1}}{\cos\delta} = \frac{\frac{c^n \cdot \sin\beta \cdot \cos\delta - b^n \cdot \sin\delta \cdot \cos\beta}{\cos\beta \cdot \cos\delta}}{\frac{\sin\beta}{\cos\beta}}$$

So,

$$\frac{a^n \cdot \cos\delta - \ \cdot b^{n-1}}{\cos\delta} = \left(\frac{c^n \cdot \sin\beta \cdot \cos\delta - \ ^n \cdot \sin\delta \cdot \cos\beta}{\cos\beta \cdot \cos\delta}\right) \cdot \frac{\cos\beta}{\sin\beta},$$

whence

$$\frac{a^n \cdot \cos\delta - \ \cdot b^{n-1}}{\cos\delta} = \frac{c^n \cdot \sin\beta \cdot \cos\delta - \ ^n \cdot \sin\delta \cdot \cos\beta}{\sin\beta \cdot \cos\delta},$$

then

$$a^n \cdot \cos\delta - a \cdot b^{n-1} = \frac{c^n \cdot \sin\beta \cdot \cos\delta - \ ^n \cdot \sin\delta \cdot \cos\beta}{\sin\beta},$$

hence

$a^n \cdot \sin\beta \cdot \cos\delta - a \cdot b^{n-1} \cdot \sin\beta = c^n \cdot \sin\beta \cdot \cos\delta - b^n \cdot \sin\delta \cdot \cos\beta.$

And, from the hypothesis $a^n = b^n + c^n$ (1), then,

$(b^n + c^n) \cdot \sin\beta \cdot \cos\delta - a \cdot b^{n-1} \cdot \sin\beta = c^n \cdot \sin\beta \cdot \cos\delta - b^n \cdot \sin\delta \cdot \cos\beta,$

or

$b^n \cdot \sin\beta \cdot \cos\delta - a \cdot b^{n-1} \cdot \sin\beta = -b^n \cdot \sin\delta \cdot \cos\beta.$

So,

$b^n \cdot (\sin\beta \cdot \cos\delta + \sin\delta \cdot \cos\beta) = a \cdot b^{n-1} \cdot \sin\beta,$

thus

$b^n \cdot \sin(\beta + \delta) = a \cdot b^{n-1} \cdot \sin\beta,$

or yet

$b \cdot \sin(\beta + \delta) = a \cdot \sin\beta.$

Therefore,

$$\frac{a}{\sin(\beta+\delta)} = \frac{b}{\sin} \quad (20)$$

However, from $\triangle (a, b, c)$, Fig. 1, and by the law of sines, the outcome is:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\delta} \ ,$$

where

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \quad (21)$$

Comparing (20) with (21), it results:

$$\frac{a}{\sin(\beta+\delta)} = \frac{a}{\sin\alpha} \ \rightarrow \ \sin\alpha = \sin(\beta+\delta) \ \therefore \ \alpha = \beta + \delta.$$

And, since $\alpha + \beta + \delta = 180°$; therefore, $2\beta + 2\delta = 180°$; or, yet, $\beta + \delta = 90°$. When ce, $\alpha = 90°$.

As consequence, it is mandatory that the triangle $\triangle (a, b, c)$, Fig. 1, is a rectangle triangle, which implies:

$$a^2 = b^2 + c^2 \quad (22)$$

And, in these terms, the equation (22) is equivalent to the equation (1). It implies that the exponentes of both equations are the same, with $n = 2$.

Finally, remains proved that for the situation 3, where $(a, b, c, n) \in \mathbb{N}$ and $a^n = b^n + c^n$ (1), never will be possible if $n > 2$.

CONCLUSION

Acknowledging that $a^n = b^n + c^n$ (1) will have three, and only three, possibilities for the position of the point *P*, that separates b^n from c^n on a^n as shown at Fig. 1, then, it implies in the three situations investigated at the argument and proof section. The situations 1, 2, and 3 were able to prove the truthiness of Fermat´s Last Theorem. For the situation 1, it is proved the nonexistence of (1), because $a^n > b^n + c^n$ when $n > 2$. About the situation 2, under the same conditions foreseen by the hypotheses, it is repeated the nonexistence of (1). That is $a^n > b^n + c^n$, whenever $n > 2$.

At last, on the analysis of situation 3, we were able to prove the only one outcome possible is $n = 2$.

Therefore, and as seen, there is nothing else to prove. Q.E.D.

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