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SOME RESULTS ON SUBDIVISION GRAPHS

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ABSTRACT

In this paper we introduce the concept of subdivision number and total subdivision number of a graph G, obtain bounds for these parameters and determine their exact values for several classes of graphs.

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INTRODUCTION

By a graph G=(V,E) we mean a finite undirected graph without loops or multiple edges. Terms defined here are used in the sense of Harary [1972]. A subset S of V is called a dominating set of G if every vertex not in S is adjacent to some vertex in S. The domination number γ(G)(or γ for short) of G is the minimum cardinality taken over all dominating sets of G. A subset S of V is called a total dominating set of G if every vertex in V is adjacent to some vertex in S. The cardinality of a smallest total dominating set of G is called the total domination number of G and is denoted byγ_t. A dominating set S of a graph G is called an independent dominating set of G if <S> is independent in G. The cardinality of a smallest independent dominating set of G is called the independent number of G and is denoted byγ_i A subdivision of an edge e=uv of a graph G is the replacement of the edge e by a path (u,v,w). The graph obtained from a graph G by subdividing each edge of G exactly once is called the subdivision graph of G and is denoted by S(G).

We need the following theorems.

Theorem 1.1[Walikar et al., 1979, p.137] Let G be a connected graph of order p$\geq$2. Then γ =p/2 if and only if G=C_4 or H^+ for some connected graph H.

Theorem 1.2 (Ore, 1962) For any graph G of order p that has no isolated vertex, $\gamma \leq \left[\frac{p}{2}\right]$

Theorem 1.3 (Arumugam and Paulraj Joseph,1996) If G is a connected graph, then γ (S(G)) = γ_i(S(G)).

In this paper we consider the following problem. Given a graph G, What is the minimum number of edges to be subdivided exactly once so that the domination number of the resulting graph exceeds that of G.

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MAIN RESULTS

Theorem

Let G be a connected graph of order $p \geq 2$. Then

$$(i)\gamma_i\left(S(G)\right) \geq \left\lceil \frac{p}{2} \right\rceil$$

$$(ii)\gamma(G) \leq \gamma(S(G)) \text{ and equality holds if and only if } G = K_2.$$

Proof

(i) Let V(G)={v_1,v_2, ...v_p) and E(G) ={e_1,e_2,......e_q}. Let w_i be the vertex of S(G) which subdivides the edge e_i.Let D be a minimum independent dominating set of S(G). Let $D_1 = D \cap V(G)$ and $D_2 = D \cap (V(S(G)) \setminus V(G))$. If $D_2 = \emptyset$ then

$D_1 =$ V(G) and hence $\gamma_i(S(G)) = p \geq \left\lceil \frac{p}{2} \right\rceil$.

Suppose $D_1 = \emptyset$. Since each vertex of D dominates exactly two vertices of V(S(G))\D, it follows that $\left|V(S(G)) \setminus D\right| \leq 2\left|D\right|$.

Hence $p + q - \left|D\right| \leq 2\left|D\right|$ so that $\left|D\right| \geq \frac{2p-1}{3} \geq \left\lceil \frac{p}{2} \right\rceil$.

Suppose both D_1 and D_2 are non-empty. Since each vertex of D_2 dominates exactly two vertices of V(G), it follows that $p - \left|D_1\right| \leq 2\left|D_2\right|$. Hence $p \leq \left|D1\right| + 2\left|D_2\right| \leq 2\left|D\right|$. Thus $\gamma_i(S(G)) = \left|D\right| \geq p/2$ so that $\gamma_i(S(G)) \geq \left\lceil \frac{p}{2} \right\rceil$.

(ii) Let D be a minimum dominating set of S(G). Let $w_i \in D$. Then at least one end of each e_i, say $u_i$does not belong to D. We now replace D by $\backslash\{w_i\} \cup \{u_i\}$. By repeating this process, we obtain a subset D_1 of V(G) such that $|D_1| \leq |D|$. Clearly $\gamma(G) \leq \gamma(S(G))$. For G = K_2, we have γ (G) = γ (S(G)) = 1.

Now, let G be connected graph with γ (G) = γ (S(G)). By Theorem 1.3, γ (G) = γ_i (S(G)) and hence it follows from (i) that $\gamma(S(G)) \geq \left\lceil \frac{p}{2} \right\rceil$. Also by Theorem 1.2 $\gamma(G) \leq \left\lfloor \frac{p}{2} \right\rfloor$.Hence $\gamma(G) \leq \left\lfloor \frac{p}{2} \right\rfloor \leq \left\lceil \frac{p}{2} \right\rceil \leq \gamma(S(G))$. Since $\gamma(G) = \gamma(S(G))$, $\left\lfloor \frac{p}{2} \right\rfloor = \left\lceil \frac{p}{2} \right\rceil$ so that p is even and $\gamma(G) = \gamma(S(G)) = p/2$. It follows from Theorem 1.1, G = C_4 or H^+ for some connected graph H. Since γ (G) = γ (S(G)), G ≠ C_4. If G = H^+ for some connected graph H of order ≥ 2, then γ_i $(S(G)) > p/2$. Hence it follows that H= K_1 and G = H^+ = K_2.

It follows from Theorem 2.1 that $\gamma(S(G)) > \gamma(G)$ for any connected graph G of order at least 3. Hence the following question naturally arises. What is the minimum number of edges to be subdivided exactly once so that the domination number of the resulting graph exceeds that of G? This motivates the following definition.

Definition

Let $G \neq K_2$ be a connected graph. The subdivision number sd(G) is defined to be the least positive integer k satisfying the following. There exists a set $S \subseteq E(G)$ with $\left|S\right| = k$ such that for the graph H obtained by subdividing each edge in S exactly once, $\gamma(H) > \gamma(G)$. We now proceed to compute sd(G) for some special classes of graphs.

Example

(i) sd(G) = 1 for any graph G with $p \geq 3$ and $\Delta = p - 1$. In particular, sd(Kp) = 1 for $p \geq 3$ and sd(Wp) = 1 for some $p \geq 3$.

(ii) Since $\gamma(Pn) = \left\lceil \frac{n}{3} \right\rceil$ and the graph obtained by subdividing edges of $P_n$is again a path, it follows that.

$$sd(Pn) = \begin{cases} 1 \text{ if } n \equiv 0(\text{mod } 3) \\ 2 \text{ if } n \equiv 2(\text{mod } 3) \\ 3 \text{ if } n \equiv 1(\text{mod } 3) \end{cases}$$

(iii) Similarly,

$$sd(Cn) = \begin{cases} 1 \text{ if } n \equiv 0(\text{mod } 3) \\ 2 \text{ if } n \equiv 2(\text{mod } 3) \\ 3 \text{ if } n \equiv 1(\text{mod } 3) \end{cases}$$

Theorem

$$sd(Km,n) = \begin{cases} 2 \text{ if } 3 \leq m \leq n \\ 3 \text{ if } m=2 \text{ and } n \geq 2. \end{cases}$$

Proof

Let $2 \leq m \leq n$. Clearly γ (Km,n) = 2. Let (X,Y) be a bipartition of Km,n with $|X| = m$ and $|Y| = n$. Let X = $\{x_1,x_2,.....,x_m)$ and Y = $\{y_1,y_2,.....,y_n)$. Let G_1 be the graph obtained from Km,n by subdividing an arbitrary edge e = uv. Clearly {u,v} is a minimum dominating set of G_1so that $\gamma(G_1) = \gamma(G) = 2$. Hence sd(Km,n) ≥ 2. We consider the following cases.

Case (i) $3 \leq m \leq n$. Let $e_1 = x_1y_1$ and $e_2 = x_2y_2$ be two independent edges of Km,n. Let G_2 be the graph obtained from Km,n by subdividing the edges e_1 ,e_2 with the vertices r,s respectively. Let S = $\{x_1, y_1, x_2, y_2, r, s\}$. Let D_2 be nay minimum dominating set for G_2. Clearly

$|S \cap D_2| = 2$, then $|D_2| = 3$. Hence it follows that $\gamma(G_2) > 2$ so that sd(Km,n) = 2.
Case(ii) m=2 and $n \geq 2$.

For any graph H obtained from Km,n by subdividing two arbitrary edges e_1 ,e_2 , $\{x_1,x_2)$ is a maximum dominating set of H. Hence $\gamma(H) = \gamma(G) = 2$ so that $sd(K_{2,n}) \geq 3$. Now let G_3be the graph obtained from Km,n by subdividing the edges x_1y_1, x_2y_2 and x_2 y_1. Clearly $\gamma(G_3) > 2$ and hence sd(Km,n) = 3.

Theorem

Let G be a connected graph with $\gamma = \frac{p}{2}$. Then sd(G) ≤ 3

Proof

By Theorem 1.1, G= C_4 or H^+ for some connected graph H. Clearly sd($C_{4)}$ = 3.
Suppose G = H^+ for some connected graph H. Let u and v be any two adjacent vertices of H. Let u_1 and v_1 be the pendant vertices adjacent to u and v respectively. Let G_1 be the graph obtained from G by subdividing the edges u_1v, uv and vv_1. Clearly $\gamma(G_1) > \gamma(G)$ so that sd(G) ≤ 3.

Theorem

For any tree T of order p ≥ 3, sd(T) ≤ 3.

Proof

The result is trivial if p = 3. Suppose $p \geq 4$. If there exists a vertex u in T such that u is adjacent to two pendant vertices v and w, then for the tree T_1 obtained from T by subdividing the edge uv, we have $\gamma(T_1) = \gamma(T) + 1$ and hence sd(T) = 1. Hence we assume that each vertex of T is adjacent to at most one pendant vertex of T. In this case, T has a vertex u of degree 2 which is adjacent to exactly one pendant vertex u. Let w be the other vertex adjacent to u. Let T_1 be the tree obtained from T by subdividing the edges uv and uw with the vertices r,s respectively. If $\gamma(T_1) > \gamma(T)$, then sd(T) ≤ 2.

Suppose $(T_1) = \gamma(T)$. Now any minimum dominating set D_1 of T_1 contains r and at least one of the vertices u,s,w.

If r,u$\in D_1$ then $D_1\setminus\{r\}$ is a dominating set of T which is a contradiction. If r,s$\in D_1$, then $(D_1 \setminus \{r,s\})\bigcup\{u\}$ is a dominating se of T, which is a contradiction. Hence r,w$\in D_1$. Let $N(w)\setminus\{s\}= \{x_1,x_2,.....,x_n\}$. We now claim that there exists an edge wx_i $(1 \leq i \leq n)$ such that for the tree T_2 obtained from T_1 by subdividing wx_i, $\gamma(T_2) > \gamma(T_1)$. If this is not true,then for each $i(1 \leq i \leq n)$, either $x_i \in D_1$ or is dominated by a vertex of D_1 other than w. Hence $(D_1 \setminus \{w\})\bigcup\{s\}$ is a minimum dominating set of T_1 which contains r,s; this is also a contradiction. Hence it follows that sd(T)$\leq$3.

Corollary

If there exists a vertex of a tree T which is adjacent to at least two pendant vertices, then sd(T)=1.

Theorem

If F is a forest, then F is an induced subgraph of a tree T_1 with $sd(T_1) = 1$, a tree T_2 with $sd(T_2) = 2$ and a tree T_3 with $sd(T_3) = 3$.

Proof

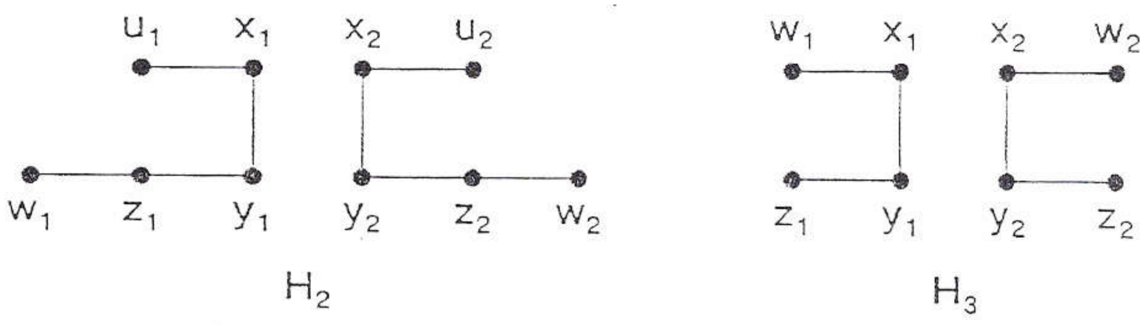
Let u be the central vertex of a path of order 3. From each component of F, select one vertex and introduce an edge joining that vertex and u. The resulting tree T_1 contains F as an induced subgraph and has a vertex namely u adjacent with two end vertices.

By Corollary 2.7, $sd(T_1) = 1$.

We now prove the existence of trees T_2 and T_3 with $sd(T_2) = 2$ and $sd(T_3) = 3$ such that T_2 and T_3 contain F as an induced subgraph.We proceed by induction on the order p of F. The claim is easily verified for p = 2. Assume that the claim is true for every forest of order p.Let F be a forest of order p + 1. Suppose $F = \overline{K}_{p+1}$. Let T_2 be the path on n vertices where n $\geq$ 2p + 1 and n $\equiv$ 2(mod 3). Let T_3 be the path on n vertices where n $\geq$ 2p + 1 and n $\equiv$ 1(mod 3). Clearly both T_2 and T_3 contain F as an induced forest and it follows from (ii) of Example 2.3, $sd(T_2) = 2$ and $sd(T_3) = 3$. Suppose now that F is non-empty. Let u be an end vertex of F and let v be adjacent to u. By induction hypothesis, there exist trees S_2 and S_3 both containing $F' = F \setminus \{u\}$ as an induced forest and $sd(S_2) = 2$ and $sd(S_3)=3$. We now construct trees T_2 and T_3 as follows. Let H_2 be the union of two paths of order 5, H_3 be the union of two paths of order 4 and label the vertices of H_2, H_3 as in Figure 2.1

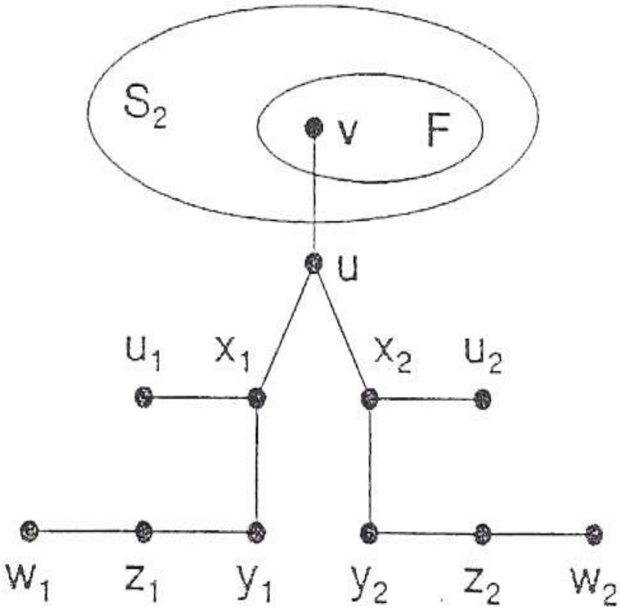
Figure

Let T_i be the tree obtained by taking the union of S_i and H_i and adding the vertex u together with the edges uv, ux_1 and ux_2(i = 2,3). Clearly T_2 , T_3 contain F as an induced forest.

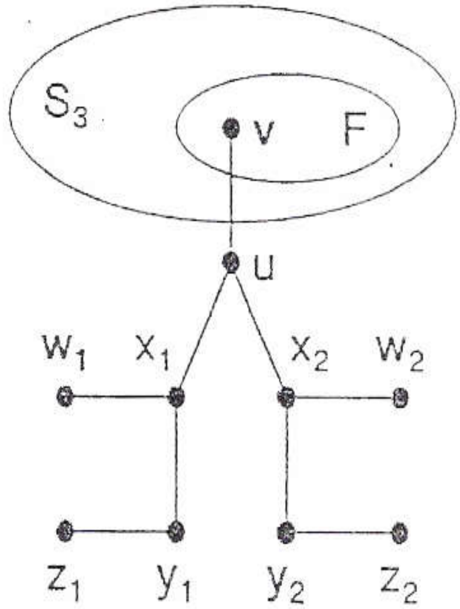


Figure

Any minimum dominating set of T_i is of the form $D_i \cup X_i$ where D_i is a minimum dominating set of S_i and X_i is a minimum dominating set of the subgraph of T_i induced by $V(H_i) \cup \{u\}$ (i = 2,3). Let J_i be the subgraph induced by the vertices in $V(H_i) \cup \{u,v\} (i = 2,3)$. Since $sd(S_i) = i$ and $sd(J_i) = i$(i = 2,3), it follows that $sd(T_i) = i$ (i = 2,3). The determination of exact value or tight bound for sd(G) for any arbitrary graph remains open. In this connection we conjecture that sd(G)$\leq$3 for any graph G. We now proceed to extend the concept of subdivision number with respect to total domination.



T2



T3

Lemma

For any connected graph G, $\gamma_t \ (S(G)) \geq p.$

Proof

Let D be any total dominating set of S(G). Let $D_1 = D \cap V(G)$ and $D_2 = D \cap (V(S(G)) \setminus V(G)).$ Since D is a total dominating set of S(G), $D_1 \neq \phi$, $D_2 \neq \phi$ and each element of D_2 dominates at most one vertex of V(G). Hence it follows that $|D_2| \geq p - |D_1|.$ Then $|D| = |D_1| + |D_2| \geq p$ so that $\gamma_t \ (S(G)) \geq p.$

Lemma

Leg G be a graph without isolated vertices. Then $\gamma_t(G) = \gamma_t(S(G))$ if and only if G = mK_2.

Proof

Let G be a graph without isolated vertices and $\gamma_t(G) = \gamma_t(S(G)) \geq p.$ It follows from Lemma 2.9, $\gamma_t(G) = \gamma_t(S(G)) \geq p.$ Hence $\gamma_t(G)$ =p so that G = mK_2.The converse is obvious.

Definition

Let G $\neq K_2$ be a connected graph. The total sub-division number tsd(G) of G is defined to be the least positive integer k satisfying the following. There exists a set $S \subseteq E(G)$ with $|S| = k$ such that for the graph H obtained by subdividing each edge in S exactly once, $\gamma_t \ (H) > \gamma_t \ (G).$

Example (i) Since

$$\gamma_t(Pn) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 (\text{mod } 4) \\ 1 + 2\left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \equiv 1 (\text{mod } 4) \\ 2 + 2\left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \equiv 2 \text{ or } 3 \ (\text{mod } 4) \end{cases}$$

and the graph obtained by subdividing the edges of Pn is again a path, it follows that

$$tsd(Pn) = \begin{cases} 1 & \text{if } n \equiv 0 \text{ or } 1 \ (\text{mod } 4) \\ 3 & \text{if } n \equiv 2 \ (\text{mod } 4) \\ 2 & \text{if } n \equiv 3 \ (\text{mod } 4) \end{cases}$$

$$tsd(Cn) = \begin{cases} 1 & \text{if } n \equiv 0 \text{ or } 1 \ (\text{mod } 4) \\ 0 & \text{if } n \equiv 2 \ (\text{mod } 4) \\ 2 & \text{if } n \equiv 3 \ (\text{mod } 4) \end{cases}$$

Theorem

For any graph G with $\Delta = p-1, tsd(G) = 2.$

Proof

Clearly $\gamma_t\ (G) = 2.$ Let u be a vertex of G with deg u = p – 1. Let G_1 be the graph obtained from G by subdividing any arbitrary edge e of G. If e passes through u, then $\{u, w\}$ where w is the vertex which subdivides e is a total dominating set of G_1; otherwise $\{u, v\}$ where v is a vertex incident with e is a total dominating set of G. Thus $tsd(G) > 1.$ Now if G_2 is the graph obtained from G buy subdividing two edges passing through u, $\gamma_t(G_2) > 2$. Hence tsd(G)=2.

Corollary

$tsd(Kp) = tsd(Wp) = 2.$

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