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REVIEW ARTICLE

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HAMILTONIAN MECHANICAL SYSTEMS ON PARA-QUATERNIONIC $Kähler$ MANIFOLDS USING FRAME FIELDS AND CO-FRAME FIELDS

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ABSTRACT

In the Hamiltonian Mechanical Systems On Para-Quaternionic $Kähler$ Manifolds we use canonical local basis $\{F^*, G^*, H^*\}$. In this paper using frame fields $i_X = X^{an+} \frac{\partial}{\partial x_{an+i}}$, $a = 0\,,1\,,2\,,3$ instead of the Hamiltonian vector field in the Hamiltonian Mechanical Systems On Para-Quaternionic $Kähler$ Manifolds using frame fields and co-frame fields we verified the generalized form the Hamiltonian equation which is in conformity with the results that have obtained previously.

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INTRODUCTION

Modern differential geometry plays an important a role to explain the dynamics of Lagrangians. So , if Q is an m-dimensional configuration manifold and $L : TQ \rightarrow R$ is a regular Lagrangian function , then it is well-known that there is a unique vector field ξon TQ such that dynamics equations is given by:

$$i_\xi \Phi_L = dE_L \qquad \rightarrow \qquad (1)$$

where Φ_L indicates the symplectic form. The triple (TQ , Φ_L , ξ) is called Lagrangian system on the tangent bundle. Also, modern differential geometry provides a good framework in which develop the dynamics of Hamiltonians. Therefore, if Q is an m-dimensional configuration manifold and $H : T^*Q \rightarrow R$ is a regular Hamiltonian function, then there is a unique vector field X on T^*Q such that dynamic equation are given by

$$i_X \Phi = dH \qquad \rightarrow \qquad (2)$$

Where Φ indicates the symplectic form. The triple (T^*Q , Φ , X) is called Hamiltonian system on the cotangent bundle T^*Q.

Nowadays, there are many studies about Lagrangian and Hamiltonian dynamics, mechanics, formalisms, system and equations [1, 2, 3, 4, 5, 6] and there in. There is real, complex, paracomplex and other analogues. As we know it is possible to produce different analogues in different spaces. Quaternions were invented by Sir William Rowan Hamiltonian as an extension to the complex numbers. Hamiltonian's defining relation is most succinctly written as:

$i^2 = j^2 = k^2 = ijk = -1$

If it is compared to the to the calculus of vectors, quaternions have slipped into the realm of obscurity. They do however still find use in the computation of rotations. A lot of physical laws in classical, relativistic, and quantum mechanics can be written pleasantly by means of quaternions. Some physicists hope they will find deeper understanding of the universe by restating basic principles in terms of quaternion algebra [7] .

The algebra B of split quaternions is a four-dimensional real vector space with basis $\{1, i, s, t\}$ given by

$i^2 = -1, s^2 = 1 = t^2, is = t = -si.$

This carries a natural indefinite inner product given by $\langle p, q\rangle = Re\overline{p}q$, where $p = x + iy + su + tv$ has $\overline{p} = x - iy - su - tv$. We have $\|p\|^2 = x^2 + y^2 - s^2 - t^2$, so a metric of signature (2,2) .This norm is multiplicative , $\|pq\|^2 = \|p\|^2\|q\|^2$, but the presence of elements of length zero means that B contains zero divisors. The fundamental structures $1, i, s, t$ are not the only split quaternions with square ± 1 . Using the multiplication rules for , one can calculate $p^2 = -1$ if and only if $p = iy + su + tv, y^2 - s^2 - t^2 = 1$, $p^2 = +1$ if and only if $p = iy + su + tv, y^2 - s^2 - t^2 = -1$ or $p = \pm 1$.

The right B-module $B^n \cong R^{4n}$ inherits the inner product $\langle \xi, \eta\rangle = Re\xi^{-T}\eta$ of signature $(2n, 2n)$. The automorphism group of $(B^n, \langle .,. \rangle)$ is $Sp(n, B) = \{A \in M_n(B): A^{-T}A = 1\}$ which is a Lie group isomorphic to $Sp(2n, R)$, the symmetries of a symplectic vector space (R^{2n}, ω).

Especially , $Sp(1, B) \cong SL(2, R)$ is the pseudo-sphere of $B = R^{2,2}$. The Lie algebra of $Sp(n, B)$ is $Sp(n, B) = \{A \in M_n(B): A + A^{-T} = 0\}$ $, so$ $Sp(1, B) = ImB$. The group $Sp(n, B) \times Sp(1, B)$ acts on B^n via :

$$(A, p).\xi = A\xi\overline{p} \quad \rightarrow \quad (3)$$

For detail see [8].

It is well-known that quaternions are useful for representing rotations in both quantum and classical mechanics.

Preliminaries: Therefore, in the present paper, we present, equations related to Hamiltonian mechanical systems on para-quaternionic $K\ddot{a}hler$ manifold. Throughout this paper, all mathematical objects and mappings are assumed to be smooth , i.e. infinitely differentiable and Einstein convention of summarizing is adopted. $\mathcal{F}(M), \chi(M)$ and $\Lambda^1(M)$ denote the set of functions on M , the of vector fields on M and the set of $1 - forms$ on M , respectively.

Theorem:

Let f be differentiable ϕ, ψ are $1 - form$, then :

- $d(f\phi) = df \wedge \phi + fd\phi$
- $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$

Frame Fields and co-Frame Fields

If U, X is a chart on smooth $n - manifold$ then written $X = (X^1, X^2, \ldots, X^n)$ we have vector fields defined on U by

$$\frac{\partial}{\partial x^i} : \wp \rightarrow \left.\frac{\partial}{\partial x^i}\right|_{\wp}$$

Such that the together the $\frac{\partial}{\partial x^i}$ form a basis at each tangent space at point in U.

We call the set of fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ a holonomic frame field over . If X is a vector field defined on some set in cluding this local chart domain U then for some smooth function X^i defined on U we have :

$$X(\wp) = \sum X^i(\wp)\left.\frac{\partial}{\partial x^i}\right|_{\wp}$$

Notice also that $dx^i : \wp \rightarrow \left.dx^i\right|_{\wp}$ defines a field of Co-vectors such that $dx^1|_{\wp}, \ldots, dx^n|_{\wp}$ forms basis of $T^*_{\wp}M$ for each $\wp \in U$. The fields form what is called a holonomic Co-frame over U. In fact , the function X^i are given by $dx^i(X) : \wp \rightarrow \left.dx^i\right|_{\wp}(X_{\wp})$. [9]

Para-Quaternionic $K\ddot{a}hler$ Manifolds:

Here, we recall hyper symplectic manifolds and para-quaternionic $K\ddot{a}hler$ manifolds given in [8].

Let $m = 4n$ identity R^{4n} $with$ B^n and consider $\dot{G} = Sp(n, B) \subset GL(4n, R). An\, Sp(n, B)$. Structure $Sp_B(M) on\, M$ defines a metric g of signature $(2n, 2n) by$ $g(u(v), u(w)) = \langle v, w\rangle$. The right action of i, s and t on B^n define endomorphism’s F , G and H of T_xM satisfying:

$$F^2 = -I, G^2 = H^2 = I, FG = H = -GF \quad \rightarrow \quad (4)$$

And the compatibility equations, for $X, Y \in T_xM$

$$g(FX,FY) = g(X,Y), g(GX,GY) = -g(X,Y) = g(HX,HY) \quad \rightarrow \quad (5)$$

Where I denotes the identity tensor of type$(1,1)$ in , and g is Riemann metric. Using (4) we obtain three 2-forms $\omega_F, \omega_G, and\ \omega_H$ given by

$$\omega_F(X,Y) = g(FX,Y),\ \omega_G(X,Y) = g(GX,Y),\ \omega_H(X,Y) = g(HX,Y).$$

The manifold M is said to be hyper symplectic if the 2-fprms $\omega_F, \omega_G, and\ \omega_H$ are all closed:

$$d\omega_F = 0, d\omega_G = 0\ and\ d\omega_H = 0.$$

Now we think of the larger structure group $Sp(n,B)Sp(1,B)$ acting on $B^n = R^{4n}$ via (3). Again we have metric of neutral signature $(2n,2n)$, but now we can not distinguish the endomorphism's $F, G\ and\ H$. Instead we have a bundle $G$of endomorphism's of TM that locally admits a basis $\{F,G,H\}$ satisfying (4) and (5). $\{F,G,H\}$ is called a canonical local basis of the bundle V in any coordinate neighborhood $U\ of\ M$. Then V is called a para-quaternionic structure in M. The pair (M,V) denotes a para-quaternionic manifold with V. A para-quaternionic manifold M is of dimension $= 4n(n \geq 1)$. A para-quaternionic structure $V$with such a Riemannian metric g is called a para-quaternionic metric structure. A manifold M with a para-quaternionic metric structure $\{g,V\}$ is called a para-quaternionic metric manifold.

The triple (M,g,V) denoted a para-quaternionic metric manifold. If $n > 1$, we say that M is para-quaternionic $K\ddot{a}hler$ if its holonomy lies in $Sp(n,B)Sp(1,B)$.

Let $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$, $i = \overline{1,n}$ be a real coordinate system on a neighborhood U of M.

The frame field represents the natural bases over R of tangent space $T(M)of\ M$ and can be written:

$$\left\{\frac{\partial}{\partial x_{an+i}}\right\}, \ a = 0,1,2,3 \quad \rightarrow \quad (6)$$

The Co-frame field represents the natural bases over $R$of the cotangent space $T^*(M)of\ M$ and can be written:

$$\{dx_{an+i}\}, \ a = 0,1,2,3 \quad \rightarrow \quad (7)$$

Taking into consideration (4), then we can obtain the expressions as follows:

$$\begin{aligned}
&F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{n+i}}, \quad G\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{2n+i}}, \quad H\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_{3n+i}}\\
&F\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_i}, \quad G\left(\frac{\partial}{\partial x_{n+i}}\right) = -\frac{\partial}{\partial x_{3n+i}}, H\left(\frac{\partial}{\partial x_{n+i}}\right) = \frac{\partial}{\partial x_{2n+i}} \quad \rightarrow \quad (8)\\
&F\left(\frac{\partial}{\partial x_{2n+i}}\right) = \frac{\partial}{\partial x_{3n+i}}, \quad G\left(\frac{\partial}{\partial x_{2n+i}}\right) = \frac{\partial}{\partial x_i}, \quad H\left(\frac{\partial}{\partial x_{2n+i}}\right) = \frac{\partial}{\partial x_{n+i}}\\
&F\left(\frac{\partial}{\partial x_{3n+i}}\right) = -\frac{\partial}{\partial x_{2n+i}}, G\left(\frac{\partial}{\partial x_{3n+i}}\right) = -\frac{\partial}{\partial x_{n+i}}, H\left(\frac{\partial}{\partial x_{3n+i}}\right) = \frac{\partial}{\partial x_i}
\end{aligned}$$

A canonical local basis $\{F^*, G^*, H^*\}\ of\ V^*$ of the cotangent space $T^*(M)$ of manifold M satisfies the condition as follows:

$$F^{*^2} = -I, G^{*^2} = H^{*^2} = I, F^*G^* = H^* = -G^*F^* \quad \rightarrow \quad (9)$$

Defining by:

$$\begin{aligned}
&F^*(dx_i) = dx_{n+i}, \quad G(dx_i) = dx_{2n+i}, \quad H^*(dx_i) = dx_{3n+i}\\
&F^*(dx_{n+i}) = -dx_i, \ G^*(dx_{n+i}) = -dx_{3n+i}, H^*(dx_{n+i}) = dx_{2n+i} \quad \rightarrow \quad (10)\\
&F^*(dx_{2n+i}) = dx_{3n+i}, G^*(dx_{2n+i}) = dx_i, \quad H^*(dx_{2n+i}) = dx_{n+i}\\
&F^*(dx_{3n+i}) = -dx_{2n+i}, G^*(dx_{3n+i}) = -dx_{n+i}, H^*(dx_{3n+i}) = dx_i
\end{aligned}$$

Hamiltonian Mechanical Systems: Here, we present Hamiltonian equations and Hamiltonian mechanical systems for quantum and classical mechanics constructed on para-quaternionic $K\ddot{a}hler$ manifold (M,g,V^*). Firstly, let (M,g,V^*) be a para-quaternionic $K\ddot{a}hler$ manifold. Suppose that an element of para-quaternionic structure V^*, a Liouville form and a 1-form on para-quaternionic $K\ddot{a}hler$ manifold (M,g,V^*) are shown by $F^*, \lambda_{F^*}\ and\ \omega_{F^*}$, respectively.

Consider:

$$\omega_{F^*} = \frac{1}{2}(x_i dx_i + x_{n+i}dx_{n+i} + x_{2n+i}dx_{2n+i} + x_{3n+i}dx_{3n+i}) \quad \rightarrow \quad (11)$$

In this equation can be concise manner:

$$\omega_{F^*} = \frac{1}{2}\sum_{a=0}^{3} x_{an+i}dx_{an+i} \quad \rightarrow \quad (12)$$

Then we have:

$$\lambda_{F^*} = F^*(\omega_{F^*}) = \frac{1}{2}(x_i dx_{n+i} - x_{n+i}dx_i + x_{2n+i}dx_{3n+i} - x_{3n+i}dx_{2n+i}) \tag{13}$$

It is concluded that if Φ_{F^*} is a closed para-quaternionic $K\ddot{a}hler$ form on para-quaternionic $K\ddot{a}hler$ manifold (M, g, V^*), then Φ_{F^*} is also a symplectic structure on para-quaternionic $K\ddot{a}hler$ manifold (M, g, V^*). Can be written Hamiltonian vector field X associated with Hamiltonian energy H by using frame fields formula:

$$X = \sum_{a=0}^{3} X^{an+i}\frac{\partial}{\partial x_{an+i}} \quad \rightarrow$$

Then

$$\Phi_{F^*} = -d\lambda_{F^*} = dx_{n+i} \wedge dx_i + dx_{3n+i} \wedge dx_{2n+i} \quad \rightarrow \tag{14}$$

Can be written i_X by using frame fields:

$$i_X = X^{an+i}\frac{\partial}{\partial x_{an+i}} \quad , \; a = 0,1,2,3 \quad \rightarrow \tag{15}$$

If : $a = 0 \quad \Rightarrow \; i_X = X^i \frac{\partial}{\partial x_i}$

$$i_X\Phi_{F^*} = X^i\frac{\partial}{\partial x_i}.dx_{n+i}.dx_i - X^i\frac{\partial}{\partial x_i}.dx_i.dx_{n+i} + X^i\frac{\partial}{\partial x_i}.dx_{3n+i}.dx_{2n+i} - X^i\frac{\partial}{\partial x_i}.dx_{2n+i}.dx_{3n+i}$$
$$i_X\Phi_{F^*} = \Phi_{F^*}(X) = X^{n+i}.dx_i - X^i.dx_{n+i} + X^{3n+i}.dx_{2n+i} - X^{2n+i}.dx_{3n+i}$$

If : $a = 1 \quad \Rightarrow \; i_X = X^{n+i}\frac{\partial}{\partial x_{n+i}}$

$$i_X\Phi_{F^*} = X^{n+i}\frac{\partial}{\partial x_{n+i}}.dx_{n+i}.dx_i - X^{n+i}\frac{\partial}{\partial x_{n+i}}.dx_i.dx_{n+i} + X^{n+i}\frac{\partial}{\partial x_{n+i}}.dx_{3n+i}.dx_{2n+i} - X^{n+i}\frac{\partial}{\partial x_{n+i}}.dx_{2n+i}.dx_{3n+i} \Rightarrow$$
$$i_X\Phi_{F^*} = \Phi_{F^*}(X) = X^{n+i}.dx_i - X^i.dx_{n+i} + X^{3n+i}.dx_{2n+i} - X^{2n+i}.dx_{3n+i}$$

If : $a = 2 \quad \Rightarrow \; i_X = X^{2n+i}\frac{\partial}{\partial x_{2n+i}}$

$$i_X\Phi_{F^*} = X^{2n+i}\frac{\partial}{\partial x_{2n+i}}.dx_{n+i}.dx_i - X^{2n+i}\frac{\partial}{\partial x_{2n+i}}.dx_i.dx_{n+i} + X^{2n+i}\frac{\partial}{\partial x_{2n+i}}.dx_{3n+i}.dx_{2n+i} - X^{2n+i}\frac{\partial}{\partial x_{2n+i}}.dx_{2n+i}.dx_{3n+i}$$
$$i_X\Phi_{F^*} = \Phi_{F^*}(X) = X^{n+i}.dx_i - X^i.dx_{n+i} + X^{3n+i}.dx_{2n+i} - X^{2n+i}.dx_{3n+i}$$

If : $a = 3 \quad \Rightarrow \; i_X = X^{3n+i}\frac{\partial}{\partial x_{3n+i}}$

$$i_X\Phi_{F^*} = X^{3n+i}\frac{\partial}{\partial x_{3n+i}}.dx_{n+i}.dx_i - X^{3n+i}\frac{\partial}{\partial x_{3n+i}}.dx_i.dx_{n+i} + X^{3n+i}\frac{\partial}{\partial x_{3n+i}}.dx_{3n+i}.dx_{2n+i} - X^{3n+i}\frac{\partial}{\partial x_{3n+i}}.dx_{2n+i}.dx_{3n+i}$$
$$i_X\Phi_{F^*} = \Phi_{F^*}(X) = X^{n+i}.dx_i - X^i.dx_{n+i} + X^{3n+i}.dx_{2n+i} - X^{2n+i}.dx_{3n+i} \quad \rightarrow \tag{16}$$

For all $a = 0, 1, 2, 3$ we obtain equation (16) furthermore , the differential of Hamiltonian energy is obtained by:

$$dH = \frac{\partial H}{\partial x_i}dx_i + \frac{\partial H}{\partial x_{n+i}}dx_{n+i} + \frac{\partial H}{\partial x_{2n+i}}dx_{2n+i} + \frac{\partial H}{\partial x_{3n+i}}dx_{3n+i} \quad \rightarrow \tag{17}$$

In this equation can be concise manner:

$$dH = \sum_{a=0}^{3}\frac{\partial H}{\partial x_{an+i}}dx_{an+i} \quad \rightarrow \tag{18}$$

With respect to (2) , if equaled (16) and (17) , the Hamiltonian vector field is found as follows:

$$X = -\frac{\partial H}{\partial x_{n+i}}\frac{\partial}{\partial x_i} + \frac{\partial H}{\partial x_i}\frac{\partial}{\partial x_{n+i}} - \frac{\partial H}{\partial x_{3n+i}}\frac{\partial}{\partial x_{2n+i}} + \frac{\partial H}{\partial x_{2n+i}}\frac{\partial}{\partial x_{3n+i}} \quad \rightarrow \tag{19}$$

Assume that a curve

$$\propto: I \subset R \rightarrow M$$

Be an integral curve of the Hamiltonian vector field X, i.e. ,

$$X(\alpha(t)) = \dot{\alpha} \quad t \in I \quad \rightarrow \tag{20}$$

In the local coordinates , it is obtained that

$\alpha(t) = (x_i , x_{n+i} , x_{2n+i} , x_{3n+i})$

And

$$\dot{\alpha}(t) = \frac{dx_i}{dt}\frac{\partial}{\partial x_i} + \frac{dx_{n+i}}{dt}\frac{\partial}{\partial x_{n+i}} + \frac{dx_{2n+i}}{dt}\frac{\partial}{\partial x_{2n+i}} + \frac{dx_{3n+i}}{dt}\frac{\partial}{\partial x_{3n+i}} \quad \rightarrow \quad (21)$$

Taking (20) if we equal (19) and (21) it holds

$$\frac{dx_i}{dt} = -\frac{\partial H}{\partial x_{n+i}} , \frac{dx_{n+i}}{dt} = \frac{\partial H}{\partial x_i} , \frac{dx_{2n+i}}{dt} = -\frac{\partial H}{\partial x_{3n+i}} , \frac{dx_{3n+i}}{dt} = \frac{\partial H}{\partial x_{2n+i}} \quad \rightarrow \quad (22)$$

Hence , the equations introduced in (22) are named Hamiltonian equations with respect to component F^* of para-quaternionic structure V^*on para-quaternionic $Kähler$ manifold (M , g , V^*), and then the triple (M, Φ_{F^*}, X) is said to be a Hamiltonian mechanical system on para-quaternionic $Kähler$ manifold (M , g , V^*) . Secondly , let (M , g , V^*) be para-quaternionic $Kähler$ manifold. Assume that a component of para-quaternion structure V^* , a Liouville form and a 1-form on para-quaternionic $Kähler$ manifold (M , g , V^*) are denoted by G^* , λ_{G^*} *and* ω_{G^*}, respectively.

Consider:

$$\omega_{G^*} = \frac{1}{2}(x_i dx_i + x_{n+i}dx_{n+i} + x_{2n+i}dx_{2n+i} + x_{3n+i}dx_{3n+i}) \quad \rightarrow \quad (23)$$

In this equation can be concise manner:

$$\omega_{G^*} = \frac{1}{2}\sum_{a=0}^{3} x_{an+i}dx_{an+i} \quad \rightarrow \quad (24)$$

Then we have calculate:

$$\lambda_{G^*} = G^*(\omega_{G^*}) = \frac{1}{2}(x_i dx_{2n+i} - x_{n+i}dx_{3n+i} - x_{2n+i}dx_i + x_{3n+i}dx_{n+i})$$

It is well-known if Φ_{G^*} is a closed para-quaternionic $Kähler$ form on para-quaternionic $Kähler$ manifold (M , g , V^*), then Φ_{G^*} is also a symplectic structure on para-quaternionic $Kähler$ manifold (M , g , V^*).

Let X a Hamiltonian vector field related to Hamiltonian energy H and given by Eq(13).

Take into consideration:

$$\Phi_{G^*} = -d\lambda_{G^*} = dx_{2n+i} \wedge dx_i + dx_{n+i} \wedge dx_{3n+i} \quad \rightarrow \quad (25)$$

Then form Eq (15) we obtained

If : If : $a = 0 \quad \Rightarrow \quad i_X = X^i \frac{\partial}{\partial x_i}$

$$i_X\Phi_{G^*} = X^i \frac{\partial}{\partial x_i}. dx_{2n+i}. dx_i - X^i \frac{\partial}{\partial x_i}. dx_i. dx_{2n+i} + X^i \frac{\partial}{\partial x_i}. dx_{n+i}. dx_{3n+i} - X^i \frac{\partial}{\partial x_i}. dx_{3n+i}. dx_{n+i}$$

$$i_X\Phi_{G^*} = \Phi_{G^*}(X) = X^{2n+i}. dx_i - X^i. dx_{2n+i} + X^{n+i}. dx_{3n+i} - X^{3n+i}. dx_{n+i}$$

If : If : $a = 1 \quad \Rightarrow \quad i_X = X^{n+i} \frac{\partial}{\partial x_{n+i}}$

$$i_X\Phi_{G^*} = X^{n+i} \frac{\partial}{\partial x_{n+i}}. dx_{2n+i}. dx_i - X^{n+i} \frac{\partial}{\partial x_{n+i}}. dx_i. dx_{2n+i} + X^{n+i} \frac{\partial}{\partial x_{n+i}}. dx_{n+i}. dx_{3n+i} - X^{n+i} \frac{\partial}{\partial x_{n+i}}. dx_{3n+i}. dx_{n+i}$$

$$i_X\Phi_{G^*} = \Phi_{G^*}(X) = X^{2n+i}. dx_i - X^i. dx_{2n+i} + X^{n+i}. dx_{3n+i} - X^{3n+i}. dx_{n+i}$$

If : If : $a = 2 \quad \Rightarrow \quad i_X = X^{2n+i} \frac{\partial}{\partial x_{2n+i}}$

$$i_X\Phi_{G^*} = X^{2n+i} \frac{\partial}{\partial x_{2n+i}}. dx_{2n+i}. dx_i - X^{2n+i} \frac{\partial}{\partial x_{2n+i}}. dx_i. dx_{2n+i} + X^{2n+i} \frac{\partial}{\partial x_{2n+i}}. dx_{n+i}. dx_{3n+i} - X^{2n+i} \frac{\partial}{\partial x_{2n+i}}. dx_{3n+i}. dx_{n+i}$$

$$i_X\Phi_{G^*} = \Phi_{G^*}(X) = X^{2n+i}. dx_i - X^i. dx_{2n+i} + X^{n+i}. dx_{3n+i} - X^{3n+i}. dx_{n+i}$$

If : If : $a = 3 \quad \Rightarrow \quad i_X = X^{3n+i} \frac{\partial}{\partial x_{3n+i}}$

$$i_X\Phi_{G^*} = X^{3n+i} \frac{\partial}{\partial x_{3n+i}}. dx_{2n+i}. dx_i - X^{3n+i} \frac{\partial}{\partial x_{3n+i}}. dx_i. dx_{2n+i} + X^{3n+i} \frac{\partial}{\partial x_{3n+i}}. dx_{n+i}. dx_{3n+i} - X^{3n+i} \frac{\partial}{\partial x_{3n+i}}. dx_{3n+i}. dx_{n+i}$$

$$i_X\Phi_{G^*} = \Phi_{G^*}(X) = X^{2n+i}. dx_i - X^i. dx_{2n+i} + X^{n+i}. dx_{3n+i} - X^{3n+i}. dx_{n+i} \quad \rightarrow \quad (26)$$

According to Eq(2) , if we equal Eq (17) and Eq (26), it yields

$$X = -\frac{\partial H}{\partial x_{2n+i}}\frac{\partial}{\partial x_i} + \frac{\partial H}{\partial x_{3n+i}}\frac{\partial}{\partial x_{n+i}} + \frac{\partial H}{\partial x_i}\frac{\partial}{\partial x_{2n+i}} - \frac{\partial H}{\partial x_{n+i}}\frac{\partial}{\partial x_{3n+i}} \quad \rightarrow \quad (27)$$

Taking Eq (20) , Eq (21) and Eq(27) are equal , we find equations

$$\frac{dx_i}{dt} = -\frac{\partial H}{\partial x_{2n+i}} \ , \frac{dx_{n+i}}{dt} = \frac{\partial H}{\partial x_{3n+i}} \ , \ \frac{dx_{2n+i}}{dt} = \frac{\partial H}{\partial x_i} \ , \ \frac{dx_{3n+i}}{dt} = -\frac{\partial H}{\partial x_{n+i}} \quad \rightarrow \tag{28}$$

Finally , the equations found in Eq (28) are called Hamiltonian equations with respect to component G^* of para-quaternionic structure V^*on para-quaternionic $Kähler$ manifold (M , g , V^*), and then the triple (M, Φ_{G^*}, X) is said to be a Hamiltonian mechanical system on para-quaternionic $Kähler$ manifold (M , g , V^*) . Thirdly, let (M , g , V^*) be para-quaternionic $Kähler$ manifold. By H^* , λ_{H^*} *and* ω_{H^*}, we give a element of para-quaternion structure V^* , a Liouville form and a 1-form on para-quaternionic $Kähler$ manifold (M , g , V^*) respectively.

Consider:

$$\omega_{H^*} = \frac{1}{2}(x_i dx_i + x_{n+i}dx_{n+i} + x_{2n+i}dx_{2n+i} + x_{3n+i}dx_{3n+i}) \quad \rightarrow \tag{29}$$

In this equation can be concise manner:

$$\omega_{H^*} = \frac{1}{2}\sum_{a=0}^{3} x_{an+i}dx_{an+i} \qquad \rightarrow \tag{30}$$

Then we have calculate:

$$\lambda_{H^*} = H^*(\omega_{H^*}) = \frac{1}{2}(x_i dx_{3n+i} + x_{n+i}dx_{2n+i} - x_{2n+i}dx_{n+i} - x_{3n+i}dx_i)$$

We know that if Φ_{H^*} is a closed para-quaternionic $Kähler$ form on para-quaternionic $Kähler$ manifold (M , g , V^*), then Φ_{H^*} is also a symplectic structure on para-quaternionic $Kähler$ manifold (M , g , V^*). Let X a Hamiltonian vector field connected with Hamiltonian energy H and given by Eq(13).

Calculating :

$$\Phi_{H^*} = -d\lambda_{H^*} = dx_{3n+i} \wedge dx_i + dx_{2n+i} \wedge dx_{n+i} \qquad \rightarrow \tag{31}$$

Then form Eq (15) we obtained

If : If : $a = 0 \quad \Rightarrow \quad i_X = X^i \frac{\partial}{\partial x_i}$

$$i_X\Phi_{H^*} = X^i \frac{\partial}{\partial x_i}.dx_{3n+i}.dx_i - X^i \frac{\partial}{\partial x_i}.dx_i.dx_{3n+i} + X^i \frac{\partial}{\partial x_i}.dx_{2n+i}.dx_{n+i} - X^i \frac{\partial}{\partial x_i}.dx_{n+i}.dx_{2n+i}$$
$$i_X\Phi_{H^*} = \Phi_{H^*}(X) = X^{3n+i}.dx_i - X^i.dx_{3n+i} + X^{2n+i}.dx_{n+i} - X^{n+i}.dx_{2n+i}$$

If : If : $a = 1 \quad \Rightarrow \quad i_X = X^{n+i} \frac{\partial}{\partial x_{n+i}}$

$$i_X\Phi_{H^*} = X^{n+i} \frac{\partial}{\partial x_{n+i}}.dx_{3n+i}.dx_i - X^{n+i} \frac{\partial}{\partial x_{n+i}}.dx_i.dx_{3n+i} + X^{n+i} \frac{\partial}{\partial x_{n+i}}.dx_{2n+i}.dx_{n+i} - X^{n+i} \frac{\partial}{\partial x_{n+i}}.dx_{n+i}.dx_{2n+i}$$
$$i_X\Phi_{H^*} = \Phi_{H^*}(X) = X^{3n+i}.dx_i - X^i.dx_{3n+i} + X^{2n+i}.dx_{n+i} - X^{n+i}.dx_{2n+i}$$

If : If : $a = 2 \quad \Rightarrow \quad i_X = X^{2n+i} \frac{\partial}{\partial x_{2n+i}}$

$$i_X\Phi_{H^*} = X^{2n+i} \frac{\partial}{\partial x_{2n+i}}.dx_{3n+i}.dx_i - X^{2n+i} \frac{\partial}{\partial x_{2n+i}}.dx_i.dx_{3n+i} + X^{2n+i} \frac{\partial}{\partial x_{2n+i}}.dx_{2n+i}.dx_{n+i} - X^{2n+i} \frac{\partial}{\partial x_{2n+i}}.dx_{n+i}.dx_{2n+i}$$
$$i_X\Phi_{H^*} = \Phi_{H^*}(X) = X^{3n+i}.dx_i - X^i.dx_{3n+i} + X^{2n+i}.dx_{n+i} - X^{n+i}.dx_{2n+i}$$

If : If : $a = 3 \quad \Rightarrow \quad i_X = X^{3n+i} \frac{\partial}{\partial x_{3n+i}}$

$$i_X\Phi_{H^*} = X^{3n+i} \frac{\partial}{\partial x_{3n+i}}.dx_{3n+i}.dx_i - X^{3n+i} \frac{\partial}{\partial x_{3n+i}}.dx_i.dx_{3n+i} + X^{3n+i} \frac{\partial}{\partial x_{3n+i}}.dx_{2n+i}.dx_{n+i} - X^{3n+i} \frac{\partial}{\partial x_{3n+i}}.dx_{n+i}.dx_{2n+i}$$
$$i_X\Phi_{H^*} = \Phi_{H^*}(X) = X^{3n+i}.dx_i - X^i.dx_{3n+i} + X^{2n+i}.dx_{n+i} - X^{n+i}.dx_{2n+i} \rightarrow \tag{32}$$

With respect to Eq (2) we equal Eq (17) and Eq (32) we find the Hamiltonian vector field given by

$$X = -\frac{\partial H}{\partial x_{3n+i}}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_{2n+i}}\frac{\partial}{\partial x_{n+i}} + \frac{\partial H}{\partial x_{n+i}}\frac{\partial}{\partial x_{2n+i}} + \frac{\partial H}{\partial x_i}\frac{\partial}{\partial x_{3n+i}} \qquad \rightarrow \tag{33}$$

Considering Eq (20) , Eq (21) and Eq(33) are equal , we find equations

$$\frac{dx_i}{dt} = -\frac{\partial H}{\partial x_{3n+i}} \ , \frac{dx_{n+i}}{dt} = -\frac{\partial H}{\partial x_{2n+i}} \ , \ \frac{dx_{2n+i}}{dt} = \frac{\partial H}{\partial x_{n+i}} \ , \ \frac{dx_{3n+i}}{dt} = \frac{\partial H}{\partial x_i} \quad \rightarrow \tag{34}$$

In the end , the equations introduced in Eq (34) are named Hamiltonian equations with respect to element H^* of para-quaternionic structure V^*on para-quaternionic $Kähler$ manifold (M , g , V^*), and then the triple (M, Φ_{H^*}, X) is said to be a Hamiltonian mechanical system on para-quaternionic $Kähler$ manifold (M , g , V^*) .

CONCLUSION

From above, Hamiltonian mechanical systems have intrinsically been described with taking into account the basis $\{F^*, G^*, H^*\}$ of para-quaternionic structure V^*on para-quaternionic $K\ddot{a}hler$ manifold $(M\ ,g\ ,V^*)$. The paths of Hamilton vector field X on the para-quaternionic $K\ddot{a}hler$ manifold are the solutions Hamiltonian equations raised in Eq (22) , Eq (28) and Eq (34), and obtained by a canonical local basis $\{F^*, G^*, H^*\}$ of vector bundle V^* on para-quaternionic $K\ddot{a}hler$ manifold $(M\ ,g\ ,V^*)$.

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