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HIGH DIMENSIONAL PAINLEVÉ INTEGRABLE SCHWARZIAN BOUSSINESQ MODELS

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ABSTRACT

The known integrable models possess Schwarzian forms with M¨obious transformation invariance, it may be one of the best ways to find new inte grable models starting from some suitable Schwarzian forms. In this paper, with introducing the high dimensional Schwarzian derivatives, the general (n + 1)-dimensional systems are obtained from the usual (1+1)-dimensional Schwartzian Boussinesq equation. A singularity structure analysis of the extension system is carried out and it is shown that arbitrary dimensional systems admit the Painlev´e property. The single soliton and the traveling wave solutions of the model are studied.

INTRODUCTION

Modern soliton theory is widely applied in almost all the physics fields, such as field theory, condensed matter physics, plasma physics, optics, particle and nuclear physics, etc (Shukla *et al*., 2007 and Kamchatnov, 2008). However, most of the present studies of the soliton theory and soliton applications are restricted in (1+1) and (2+1)-dimensions due to lacking of known higher dimensional integrable systems. It is significant for nonlinear physics to find the high dimensional integrable systems. According to the fact that almost all the known integrable models can be transformed to the Schwarzian forms, some quite general conformal invariant equations in arbitrary dimension are found to be integrable under the meaning that they possess the Painlev´e property [3, 4, 5, 6, 7, 8, 9]. In this letter, the (n+1)-dimensional Painlev´e integrable Boussinesq systems are constructed with selecting the high dimensional Schwarzian derivatives. The single soliton and traveling wave solutions of the systems are obtained.

(n+1)-dimensional Boussinesq extension

The Schwartzian Boussinesq equation form is (Weiss *et al*., 1983).

$$\left(\frac{\phi_t}{\phi_x}\right)_t + \frac{1}{2}\left(\frac{\phi_t^2}{\phi_x^2}\right)_x + \frac{1}{3}\{\phi; x\}_x = 0, \qquad (1)$$

where ϕ satisfy an equation formulated in terms of Schwartzian derivative

$$\{\phi; x\} = \left(\frac{\phi_{xx}}{\phi_x}\right)_x - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)^2. \qquad (2)$$

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In order to get the high dimensional systems, the high Schwartzian derivative is introduced (Toda and Yu, 2000)

$$S_{2+1}[\phi;x,y]=\left(\frac{\phi_{xx}}{\phi_x}\right)_y-\frac{1}{2}\partial_x^{-1}\left(\frac{\phi_{xx}^2}{\phi_x^2}\right)_y, \qquad (3)$$

which is invariant in the M¨obious transformation. To extend the Schwartzian Boussinesq equation (1) in high dimension, we may take many forms. Here, we take an (n + 1)-dimensional Boussinesq extension as

$$\sum_{i=1}^{n}\left[a_i\left(\frac{\phi_t}{\phi_{x_i}}\right)_t+\frac{b_i}{2}\left(\frac{\phi_t^2}{\phi_{x_i}^2}\right)_{x_i}+\frac{c_i}{3}S_{2+1}\{\phi;x,x_i\}_x\right]=0, \qquad (4)$$

where a_i, b_i and c_i (i = 1, 2, · · · , n) are constants. (Lou, 1998) turns into the usual Boussinesq Schwartzian form with $a_i = b_i = c_i = 0$ (i = 2, 3, · · · , n). Meanwhile, (Lou, 1998) is invariant under the M¨obious transformation

$$\phi\to\frac{a+b\phi}{c+d\phi}, \qquad ad\neq bc.$$

In order to make use of the Weiss-Tabor-Carnevale (WTC) approach (Weiss, 1983), we make the following transformations

$$\phi=\exp F, \qquad u_0=F_t, \qquad u_i=F_{x_i}, \quad i=1,2,\cdots,n. \qquad (5)$$

Substituting expressions (5) into (4), we get the following system

$$\sum_{i=1}^{n}\left[a_i\left(\frac{u_{0,t}}{u_i}+\frac{u_0u_{0,x_i}}{u_i^2}\right)+b_i\left(\frac{u_0^2u_{i,x_i}}{u_i^3}+\frac{u_0u_{0,x_i}}{u_i^2}\right)+c_1\left(\frac{u_{1,x_1x_1}}{3u_1}-\frac{u_{1,x_1}^2}{2u_1^2}-\frac{1}{6}u_1^2\right)+$$
$$c_{i+1}\left(\frac{u_{1,x_1}^2u_{1,x_{i+1}}}{u_1^3}+\frac{u_{1,x_1x_1x_{i+1}}}{3u_1}-\frac{u_{1,x_1x_1}u_{1,x_{i+1}}}{3u_1^2}-\frac{u_{1,x_1}u_{1,x_1x_{i+1}}}{u_1^2}-\frac{1}{3}u_1u_{1,x_{i+1}}\right)\right]=0, \qquad (6a)$$

$$u_{i,t}-u_{0,x_i}=0, \qquad i=1,2,\cdots,n, \qquad (6b)$$

where (6b) is the compatibility condition of transformations (5). We use the standard WTC approach to check the Painlev´e property of (6). The function u_i expands

$$u_i=\sum_{j=0}^{\infty}u_{ij}\phi_1^{j+\alpha}, \qquad i=0,1,\cdots,n, \qquad (7)$$

where u_{ij} are analysis functions of (t, x_i) and α is integer to be determined. According to the leading order analysis, we obtain

$$\alpha_i=-1, \qquad u_{00}^2=\phi_{1,t}^2, \qquad u_{i0}^2=\phi_{1,x_i}^2. \qquad (8)$$

Substituting (7) and (8) into (6), we have

$$(j+1)(j-1)^{n+1}(j-2)\sum_{i=1}^{n}c_{i+1}u_{(i+1)0}=f(u_{ik},i=0,1,\cdots,n,k\le j-1), \qquad (9a)$$

$$(j-1)(u_{0j}u_{i0}-u_{ij}u_{00})=u_{i(j-1),t}-u_{0(j-1),x_i}, \qquad (9b)$$

where f is a complicated function of (u_{ik}, i = 0, 1, · · · , n, k ≤ j 1) and the derivatives of the singularity manifold ϕ 1. The resonance points are located at

$$j=-1,\underbrace{1,1,\cdots,1}_{n+1},2. \qquad (10)$$

The resonance at j = 1 corresponds to the arbitrary singularity manifold ϕ 1. At n + 1 resonances j = 1 and one resonance j = 2, there are corresponding n + 2 compatibility conditions

$$\sum_{i=1}^{n}\left(c_{i+1}\phi_{i+1}\phi_{1,x_1}^2 - c_{i+1}\phi_{i+1}u_{10}^2\right) = 0, \qquad (11a)$$

$$u_{i0,t} - u_{00,x_i} = 0, \qquad i = 1, 2, \cdots, n, \qquad (11b)$$

$$\prod_{i=1}^{n} u_{i0}\Big[c_1\left(u_{10}^2 - \phi_{1,x_1}^2\right) + \sum_{i=1}^{n} 2c_{i+1}\Big(u_{00}u_{00,x_{i+1}} - \phi_{1,x_1}\phi_{1,x_1x_{i+1}} - \phi_{1,x_1x_1}\phi_{1,x_{i+1}} + \frac{\phi_{1,x_1}\phi_{1,x_{i+1}}u_{00,x}}{u_{00}}\Big)\Big] = 0. \qquad (11c)$$

Fortunately, it is straightforward to see that the conditions (11) are satisfied identically using the results of (8). Therefore, the (n + 1)-dimensional Schwartzian Boussinesq system is integrable in the sense of the Painlev´e analysis.

Furthermore, one can prove the new generalized (n + 1)-dimensional Schwarzian Boussinesq type model

$$\sum_{i=0}^{n}\left[a_i\left(\frac{\phi_t}{\phi_{x_i}}\right)_t + \frac{1}{2}b_i\left(\frac{\phi_t^2}{\phi_{x_i}^2}\right)_{x_i} + \frac{1}{3}c_i\{\phi; x_i\}_{x_i}\right] + \frac{1}{3}\sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij}(S^{[x_ix_j]})_{x_i}$$

$$+\frac{1}{3}\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n} d_{ijk}(S^{[x_ix_jx_k]})_{x_i} = 0, \qquad (12)$$

with two and three dimensional Schwarzian derivatives (Zhang *et al*., 2002)

$$S^{[x_ix_j]} = \frac{\phi_{x_ix_ix_j}}{\phi_{x_j}} - \frac{\phi_{x_ix_i}\phi_{x_ix_j}}{\phi_{x_i}\phi_{x_j}} - \frac{1}{2}\frac{\phi_{x_ix_j}^2}{\phi_{x_j}^2},$$

$$S^{[x_ix_jx_k]} = \frac{\phi_{x_ix_jx_k}}{\phi_{x_i}} - \frac{\phi_{x_ix_i}\phi_{x_jx_k}}{\phi_{x_i}^2} - \frac{\phi_{x_ix_i}\phi_{x_j}\phi_{x_ix_k}}{\phi_{x_i}^3} - \frac{\phi_{x_ix_i}\phi_{x_k}\phi_{x_ix_j}}{\phi_{x_i}^3} + \frac{3}{2}\frac{\phi_{x_ix_i}^2\phi_{x_j}\phi_{x_k}}{\phi_{x_i}^4},$$

is Painlev´e integrable.

Exact solution for (n+1)-dimensional Boussinesq extension The investigation of the exact solutions of nonlinear evolution equations plays an important role in the study of nonlinear wave phenomena (Fan, 2002). Here, we shall study the single soliton and the traveling wave solutions of the (n+1)-dimensional Boussinesq extension system (4).

It is straightforward to see that the model (4) possesses a simple single soliton

$$\phi = a + \exp(\sum_{i=1}^{n} k_i x_i + \omega t), \qquad (13)$$

with a is constant and $c_1 = 0$. The traveling wave solution writes as

$$\phi = \phi(\xi), \qquad \xi = \sum_{i=1}^{n} kx_i - ct, \qquad (14)$$

where k and c are arbitrary constants to be determined. We can easily find that the equation (4) is fully satisfied with $c_1 = 0$, $\sum^{n}_{i} = 1$ $c_i = 0$. Due to the solution (14) including an arbitrary function, we can obtain different forms of solution with selection arbitrary function ϕ. We shall select the arbitrary function to be Jacobian elliptic and hyperbolic functions as the explicit example. The motivation behind this choice stems from the fact that the limiting forms of these functions happen to be localized functions [15]. Here, we take (2+1)-dimensional extension system as example and choose the arbitrary function

$$\phi = \mathrm{sn}(\xi, m), \qquad (15)$$

where m is the modulus of the Jacobi elliptic function. In the left panel of Fig.1, we show the solution (15) with the parameters k = c = 1 and m = 0.2. The solution ϕ is selected

$$\phi = 1 + \exp(\tanh(\xi)) + \sin(\xi). \qquad (16)$$

The corresponding solution (16) is plotted with the parameters k = c = 1 in the right panel of Fig.1. Solution (16) describes a kind of periodic-kink interaction solitary wave which has been studied (Ren, 2009 and Lou *et al*., 2005).

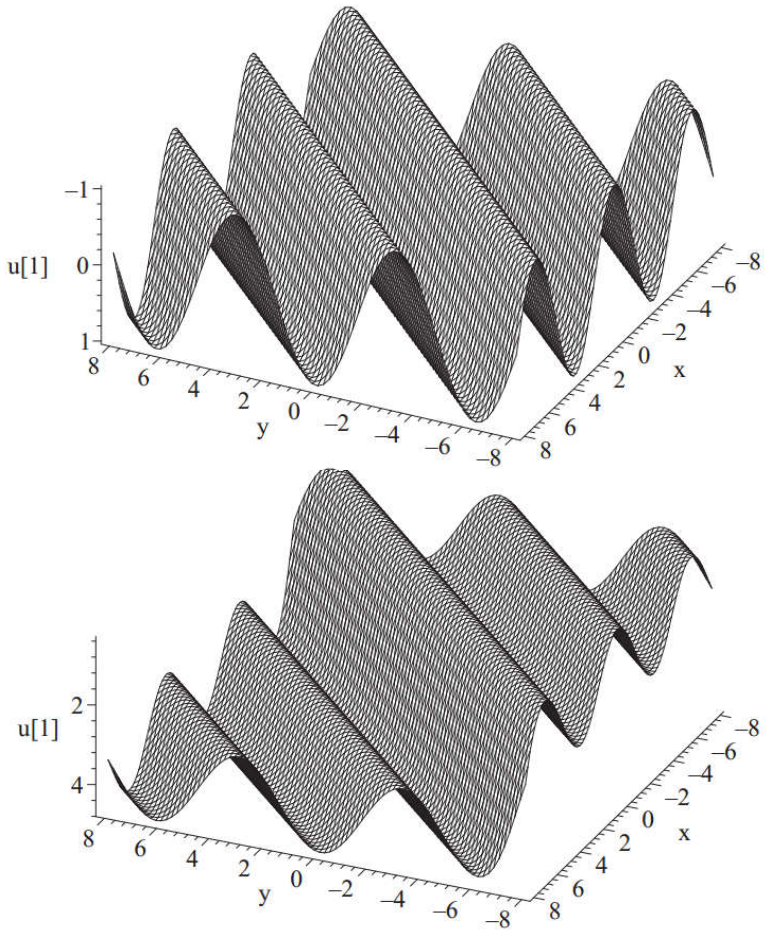

Figure 1. Evolution of the solution ϕ (15) and (16) at t = 0, respectively

Conclusion

In summary, we have extended the (1+1)-dimensional Schwartzian Boussinesq equation to the arbitrary dimensional system with selecting the high dimensional Schwarzian derivatives. We have shown that the new system satisfies the Painlev´e property and invariant under the M¨obius transformation using the standard WTC method. The single soliton and the traveling wave solutions are obtained for the (n + 1)-dimensional system (6). The properties of the exact solutions are shown by some figures. In the meanwhile, the integrable properties for the arbitrary dimensional (12) are worthy to study further.

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